PERTH MODERN SCHOOL
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Independent Public School

Mathematics Specialist Year 11

Student name: ___________________________ Teacher name: ________________

Date: Friday 21st February 2020

Task type: **Response**

Time allowed: **45 mins**

Number of questions: **7**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **45 marks**

Task weighting: **10%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 **(2.3.2)** **(3 marks)**

(a) Give a decimal representation of $\frac{2}{7}$ (1 mark)

$$7\overline{\big)\,2.{}^{2}0{}^{6}0{}^{4}0{}^{5}0{}^{1}0{}^{3}0{}^{2}0}\quad 0.\dot{2}8571\dot{4}2$$

$$\frac{2}{7} = 0.\dot{2}8571\dot{4}$$ ✓

(b) Write $0.3\dot{2}5\dot{4}$ in the form $\frac{m}{n}$ where *m* and *n* are integers and n≠0. An answer without working will not be awarded any marks (2 marks)

$$x = 0.3\dot{2}5\dot{4}$$
$$10x = 3.\dot{2}5\dot{4}$$
$$10000x = 3254.\dot{2}5\dot{4}$$ ✓
$$9990x = 3251$$
$$x = \frac{3251}{9990}$$ ✓

answer only
0 marks

Question 2 **(2.3.1)** **(3 marks)**

Answer with True or False;

(a) The product of two rational numbers can be irrational (1 marks)

false ✓

(b) The sum of two irrational numbers is always irrational (1 marks)

false ✓ consider $\sqrt{2} + (3 - \sqrt{2})$

(c) The quotient of two irrational numbers is also irrrational (if defined) (1 marks)

false ✓ consider $\frac{3\sqrt{2}}{\sqrt{2}}$

Question 3 **(2.3.1)** **(8 marks)**

(a) Assume that a is even and b is odd. Prove that ab^2 is even. (3 marks)

let $a = 2m$ and $b = 2n+1$ where $m, n \in \mathbb{Z}$

thus $ab^2 = 2m(2n+1)^2$

$= 2m(4n^2 + 4n + 1)$

$= 8mn^2 + 8mn + 2m$

$= 2(4mn^2 + 4mn + m)$ which is even

$\therefore$ If a is even and b is odd then ab^2 is even

(b) Show that $u^2 + t^2 \geq -2ut$ given that u and t are real numbers. (3 marks)

$u^2 + t^2 \geq -2ut \Rightarrow u^2 + t^2 + 2ut \geq 0$

we have $u^2 + t^2 + 2ut$

$= (u+t)^2$

≥ 0 since we are squaring a real number

$\therefore u^2 + t^2 \geq -2ut$

(c) Prove that $\frac{w}{w+3} > \frac{w-3}{w}$ given that $w \in \{1, 2, 3 \ldots\}$ (4 marks)

$$\frac{w}{w+3} > \frac{w-3}{w} \Rightarrow \frac{w}{w+3} - \frac{w-3}{w} > 0$$

we have $\frac{w}{w+3} - \frac{w-3}{w}$

$$= \frac{w^2}{w(w+3)} - \frac{(w-3)(w+3)}{w(w+3)}$$

$$= \frac{w^2 - (w^2-9)}{w(w+3)}$$

$$= \frac{9}{w(w+3)}$$ ← always positive since w is a natural number

> 0

$$\therefore \frac{w}{w+3} > \frac{w-3}{w}$$

Question 4 **(1.3.1)** **(3 marks)**

(a) Negate the statement "The sum of each pair of prime numbers is even" (1 mark)

Some pair of primes does not have an even sum

(b) Negate the statement "12 is divisible by 2 or 5". (1 mark)

12 is not divisible by 2 and 12 is not divisible by 5

(c) Write down the contrapositive of "If $2^x \geq 2^y$, then $x \geq y$" (1 mark)

if $x < y$ then $2^x < 2^y$

Question 5 **(1.3.1, 2.3.1)** **(6 marks)**

Write down the contrapositive and prove the following statements

(a) If $n^2 - 10n + 9$ is even, then n is odd. (3 marks)

Contrapositive: If n is even then $n^2 - 10n + 9$ is odd ✓

Proof: let $n = 2k$ for some $k \in \mathbb{Z}$

thus $n^2 - 10n + 9 = (2k)^2 - 10(2k) + 9$ ✓

$= 4k^2 - 20k + 9$

$= 4k^2 - 20k + 8 + 1$

$= 2(2k^2 - 10k + 4) + 1$ ✓

which is odd

$\therefore$ if $n^2 - 10n + 9$ is even, then n is odd

(b) If $2x + 3y \geq 12$, then $x \geq 3$ or $y \geq 2$ (3 marks)

Contrapositive: If $x < 3$ and $y < 2$ then $2x + 3y < 12$ ✓

Proof: If $x < 3$ and $y < 2$ then

$2x + 3y < 2(3) + 3(2)$ ✓

$< 6 + 6$

< 12 ✓ as required

$\therefore$ If $2x + 3y \geq 12$ then $x \geq 3$ or $y \geq 2$

Question 6 **(3.1.1, 3.1.4, 3.1.9)** **(9 marks)**

Use proof by contradiction for the following questions;

(a) Suppose that $5^x = 8$. Prove that x is irrational. (4 marks)

Suppose x is rational ✓

we can write $x = \frac{m}{n}$, $n \neq 0$ and $m, n \in \mathbb{N}$

thus $5^x = 8 \Rightarrow 5^{\frac{m}{n}} = 8$

$(5^{\frac{m}{n}})^n = 8^n$

$5^m = 8^n$ ✓

$\underbrace{5^m}_{\text{LHS is odd}} = \underbrace{2^{3n}}_{\text{RHS is even}}$ ✓

The above gives a contradiction ✓ $\therefore$ x is not rational

(b) Suppose that $m^2 - n^2 - 4 = 0$. Prove that m and n cannot both be natural numbers (5 marks)

Suppose both m and n are natural numbers ✓

we have $m^2 - n^2 - 4 = 0$

$\Rightarrow m^2 - n^2 = 4$

$(m+n)(m-n) = 4$ ✓

The only factors of 4 are 1, 2 and 4.

$2 \times 2 = 4$
$1 \times 4 = 4$
$4 \times 1 = 4$

Since $\underbrace{(m+n)}_{4} > \underbrace{(m-n)}_{1}$

$m + n = 4$ and $m - n = 1$ ✓

solving simultaneously by adding gives

$2m = 5$

$m = \frac{5}{2}$ ✓ (and $n = \frac{3}{2}$) both not whole

This is a contradiction ✓ $\Rightarrow$ m & n are not both natural numbers.

Question 7 **(3.1.1, 3.1.4, 3.1.9)** **(10 marks)**

(a) Write the contrapositive of "if m^3 is divisible by 3 then m is divisible by 3" (1 mark)

If m is not divisible by 3 then m^3 is not divisible by 3 ✓

(b) Prove the contrapositive above (hint: you will need to consider cases) (4 marks)

If m is not divisible by 3 then

either $m = 3k + 1$ ✓

or $m = 3k + 2$ ✓

$m = 3k$ and $m = 3k+3$ are divisible by 3

for case 1, $m = 3k + 1$

$\Rightarrow m^3 = (3k+1)^3$ CAS expand

$= 27k^3 + 27k^2 + 9k + 1$

$= 3(9k^3 + 9k^2 + 3k) + 1$ ✓

which is not divisible by 3

for case 2, $m = 3k + 2$

$m^3 = (3k+2)^3$ CAS expand

$= 27k^3 + 24k^2 + 36k + 8$

$= 3(9k^3 + 8k^2 + 12k + 2) + 2$ ✓

which is not divisible by 3

Therefore if m^3 is divisible by 3 then m is divisible by 3

(c) Hence, prove by contradiction that $\sqrt[3]{3}$ is irrational. (6 marks)

suppose $\sqrt[3]{3}$ is rational

so $\sqrt[3]{3} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$

Assume p and q have no common factors

Thus $\left(\sqrt[3]{3}\right)^3 = \left(\frac{p}{q}\right)^3$

$\Rightarrow \quad 3 = \frac{p^3}{q^3}$

$\Rightarrow \quad p^3 = 3q^3 \quad$ —— ①

$\Rightarrow$ p^3 is divisible by 3

$\Rightarrow$ p is divisible by 3 (proven above)

$\Rightarrow \quad p = 3k$ for some $k \in \mathbb{N}$

$\Rightarrow \quad (3k)^3 = 3q^3$

$\Rightarrow \quad 3q^3 = 9k^3$

$\Rightarrow \quad q^3 = 3k^3$

$\Rightarrow$ q^3 is divisible by 3

$\Rightarrow$ q is divisible by 3

Since p and q have a common factor of 3, this is a contradiction

$\therefore \sqrt[3]{3}$ must be irrational